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How to integrate over central charges

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Abstract. We construct dimensionless actions for fields which have non-trivial central charge behaviour. By means of optimisation methods for systems with constraints, we show how such actions must be defined in order to give the correct four-dimensional equations of motion. This is considered initially for a single real scalar free field with one central charge, and extended to successively more complicated situations. In particular we discuss fields with more than one central charge with suitable constraint conditions. We also consider actions for superfields with central charges and obtain full superspace actions for the degenerate cases. Fields without central charges are also described by integration over central charge dimensions. We conclude that our four-dimensional world may be embedded as the boundary of a higher-dimensional manifold whose extra dimensions are not directly accessible on-shell.

1. Introduction

The recent proofs (Mandelstam 1982, Brink *et al* 1982, Howe *et al* 1983) of the finiteness of $N=4$ super Yang–Mills theory ($N=4$ SYM) and of companion broken theories (Taylor 1983b, Rajpoot *et al* 1982, Namazie *et al* 1982, Parkes and West 1983) indicate the power of supersymmetry (SUSY) to ameliorate ultraviolet divergences of renormalisable quantum field theories. The hope that extended supersymmetric theories of gravitation, that is, extended supergravities (N-SGRs), would also be finite to all orders has yet to be realised. This is in spite of the progress in construction of these theories in terms of their physical fields (Cremmer 1982).

The essential ingredient for the analysis of ultraviolet divergences in a supersymmetric theory is well known to be a superfield version of the theory. The construction of such a version is equivalent to the discovery of the auxiliary fields of the theory, where these fields are such as to allow closure of the supersymmetry algebra but yet may be eliminated algebraically by their equations of motion. Whereas various sets of auxiliary fields have been discovered for $N=1$ and 2 SGR (Stelle and West 1978, Ferrara and von Nieuwenhuizen 1978, Fradkin and Vasiliev 1979, de Wit and van Holten 1979) there has been little progress in achieving similar success for $N\geq 3$.

The existence of a barrier to the construction of auxiliary fields for $N=3$ and 4 SGR was shown recently (Rivelles and Taylor 1981, Taylor 1982b) as well as for $N=4$ SYM (Rocek and Siegel 1981). This has now (Rivelles and Taylor 1983) been extended both to all $N\geq 3$ in four dimensions, as well as to N-SGR and N-SYM for various N in higher dimensions. The resulting no-go theorems indicate that in general auxiliary fields cannot be constructed for all interesting cases. Thus in particular the tentative conclusions (Grisaru and Siegel 1982) on the possible divergence of N-SGR assuming a full extended superfield version of these theories appears very difficult to substantiate. At the same time the no-go theorems seem to indicate the impossibility of constructing any version of N-SGR or 4-SYM in terms of fully extended superfields.

There are three methods (Taylor 1983a) which seem able to bypass the $N=3$ barrier, all depending on the reduction of the number of spinor generators in the N-SUSY algebra by a factor of 2. The first of these uses only $N/2$ superfields and one imposes the full N-supersymmetry afterwards on S-matrix elements. Such a method was used for 4-SYM in Howe *et al* (1983) to give a fully Lorentz-covariant proof of the finiteness of 4-SYM. It should also allow $N=8$ SGR to be constructed in terms of $N=4$ superfields (Bufton and Taylor 1983b), though this could only lead to finiteness up to three loops (Grisaru and Siegel 1982).

A second method uses the light-cone gauge, in which the N-supersymmetry algebra reduces to the N-SUSY light-cone subalgebra for which the auxiliary field problem disappears. This is because all spinors are reduced in dimension by a factor of 2, corresponding to elimination of the non-propagating modes of a Dirac spinor in terms of the propagating ones. Such an approach was used by Mandelstam (1982), Brink *et al* (1982), Taylor (1983b), Rajpoot *et al* (1982) and Namazie *et al* (1982). It has also been applied to N-SGR for all $N \leqslant 8$ (Taylor 1982a) to allow the construction of superfield versions of these theories (to within a controllable ambiguity). As for the $N/2$-superfield version of N-SGR, these fully locally light-cone supersymmetric versions of N-SGR cannot be proven finite to all orders of perturbation theory, so that this approach also seems unsatisfactory.

We are left with the final alternative, which is to modify the N-SUSY algebra, for $N \geqslant 3$, by the addition of central charges, Z^{ij}, Z^{ij*} ($1 \leqslant i, j \leqslant N$). These are operators which commute with all the generators of N-SUSY, but can enter to reduce the maximum spin of a SUSY-multiplet by a factor of 2 if the degeneracy condition

$$Z^{ij}Z^{*jk} = \Box\delta^i_k \tag{1.1}$$

is satisfied by all fields of the multiplet (Sohnius 1978, Taylor 1980). This spin-reducing property also evades the no-go theorems of Rivelles and Taylor (1981), Taylor (1982b) and Rivelles and Taylor (1983) by means of a Dirac condition which has been analysed elsewhere (Rands and Taylor 1983a, b) and which may be used in a covariant superspace description. If the N-SUSY generators are, in chiral notation, $S^i_{\alpha+}$, $S_{\alpha-j}$ and associated covariant derivatives $D^i_{\alpha+}$, $D_{\alpha-j}$, with $(S^i_{\alpha+})^* = S_{\alpha-i}$, $(D^i_{\alpha+})^* = D_{\alpha-i}$

$$[S^i_{\alpha+}, S^j_{\beta+}]_+ = 2\varepsilon_{\alpha+\beta+}Z^{ij}, \tag{1.2}$$

$$[S^i_{\alpha+}, S_{\beta-j}]_+ = -2(\not{p}C)_{\alpha+\beta-}\delta^i_j, \tag{1.3}$$

and similar relations for the D's (which all anticommute with the S's), the Dirac condition is

$$\not{p}S^i_{\alpha+} = Z^{ij}S_{\alpha-j}. \tag{1.4}$$

If (1.4) and its complex conjugate are satisfied on a superfield then (1.1) is satisfied, and (1.3) follows from (1.2). Only the set of generators $S^i_{\alpha+}$ are thus needed explicitly, the $S_{\alpha-i}$ being constructed in terms of them. Spin reduction has thus occurred, but moreover there are only half the number of N-SUSY generators, so that the $N=3$ barrier can be avoided (Rivelles and Taylor 1983, Taylor 1983a).

The possibility of using the algebra (1.2), (1.3) to construct N-SGR has already been considered by a number of authors. Field redefinition rules (Rivelles and Taylor

1982a) were utilised to construct tentative linearised N-SGRs for $N=4$, 6 and 8 (Taylor 1981). Linearised versions of $N=2$ SGR were constructed with two central charges (Rivelles and Taylor 1982b), as well as an earlier version with one central charge at the full nonlinear level (de Wit *et al* 1980, de Wit 1981). Earlier attempts were also made to construct $N=4$ SYM (Sohnius *et al* 1980) and $N=8$ SGR (Cremmer *et al* 1980) using multiplets for the gauge vector field or the spin-2 field, respectively, carrying central charge in contradiction to the $N=2$ models in which the $N=2$ Weyl multiplet has no central charge.

None of these analyses has allowed the construction of a superfield action. The only superfield actions that have been presented have not used a full integration over superspace but only the integration of a constrained superfield over a subspace of θ and $\bar{\theta}$ (Sohnius 1978, Taylor 1980). The difficulty in using the full superspace measure for $N=$ SUSY, $\mathrm{d}^4x\ \mathrm{d}^{2N}\theta\ \mathrm{d}^{2N}\bar{\theta}$, is that its length dimension is $(4-2N)$, which becomes progressively more negative as N increases. In order to have the N-SGR analogue of the $N=1$ SGR action, the full superspace volume (Wess and Zumino 1978)

$$\kappa^{-2}\int \mathrm{d}^4x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\, E \tag{1.5}$$

where E is the determinant of the achtbein $E_A{}^M$, it is necessary to adjoin to the space-time variables x^μ $(1\leq\mu\leq4)$ further Bose dimensions z^l, with $1\leq l\leq 2(N-1)$. Thus for $N=2$, 2 extra bosonic variables are required (Rogers 1982), whilst for $N=4$ and 8 a further 6 and 14 Bose variables are needed respectively. Thus the naive N-SGR extension of (1.5), when the required extra dimensions are to be included, is

$$\kappa^{-2}\int \mathrm{d}^4x \int \mathrm{d}^{2(N-1)}z \int \mathrm{d}^{2N}\theta\ \mathrm{d}^{2N}\bar{\theta} E \tag{1.6}$$

which might be expected to be valid for $N=1$, 2, 4 and 8 if suitable constraints are to be imposed on the $E_A{}^M$ extending those in the case of $N=1$. A similar formula

$$\int \mathrm{d}^4x\ \mathrm{d}^{2(N-1)}z \int \mathrm{d}^{2N}\theta\ \mathrm{d}^{2N}\bar{\theta}\ \mathrm{Tr}(F_{\alpha\beta})^2 \tag{1.7}$$

may also be conjectured for N-SYM. The same extra variables are also required to be able to give a dimensionless form to the action for a scalar superfield Φ, of dimension -1, of form

$$\int \mathrm{d}^4x \int \mathrm{d}^{2(N-1)} \int \mathrm{d}^{2N}\theta\ \mathrm{d}^{2N}\bar{\theta}\ \Phi^+\Phi. \tag{1.8}$$

Such an action might be hoped for as the appropriate form to describe the $N=2$ hypermultiplet (Sohnius 1978, Taylor 1980) or its $N=4$ and 8 analogues (Rands and Taylor 1983a, b). The formulae (1.6), (1.7) and (1.8) require very careful analysis before they can be expected to be used satisfactorily. In particular, we have to ask how we can interpret the extra dimensions and integration over them. They are expected to be related to the central charges Z^{ij}, but exactly how is not initially clear.

A natural explanation of the extra bosonic dimensions $z_1, \ldots, z_{2(N-1)}$ is that they are truly present in nature but only become accessible at suitably high energies such as the Planck energies. Such an interpretation is the basis of the Kaluza–Klein (KK) approach, which has recently become of renewed interest (Witten 1981, Salam and Strathdee 1981).

This use of extra variables as higher dimensions has led to an elegant construction (Cremmer 1982) of $N=8$ supergravity in $d=4$ (d denotes dimension of space–time) by trivial reduction from $N=1$ SGR in $d=11$. However, if we wish to remain in $d\leqslant 11$, as (1.6)–(1.8) would indicate, then the no-go theorems (Rivelles and Taylor 1983) are not avoided; auxiliary fields and a superfield formulation will not exist. Moreover, for $N=8$ SGR there are 14 extra Bose dimensions, not 7, so that there is a decided mismatch between the known $N=1$ SGR in $d=11$ and its extension to $d=18$. Indeed $N=1$ SGR in $d=18$ would unavoidably have fields of spin higher than two amongst its components, so would be expected to be inconsistent. We conclude that the KK interpretation is the wrong one for the extra dimensions. A similar problem arises in the recent attempt (Rogers 1982) to use the full superspace measure (1.6) for $N=2$; the equations of motion resulting appear to be in a six-dimensional space–time.

If we are not to regard the new dimensions $z_1, \ldots z_{2(N-1)}$ as ones in which we can move freely, we might turn to the opposite extreme and suppose that we can never venture off a four-dimensional submanifold Σ_4 embedded in $2(N+1)$-dimensional space–time $S_{2(N+1)}$. The question we must answer is then to explain how the integrals (1.6)–(1.8) over the whole of this $2(N+1)$-dimensional space–time can only describe dynamics on Σ_4 and not in the whole of $S_{2(N+1)}$. That is the purpose of this paper. Such a question has already been analysed in the case of one extra space variable by means of dimensional reduction by Legendre transformation by Sohnius *et al* (1980). However, we have to consider at least two extra space variables, when such a method fails completely. Instead we will show that a viable interpretation of (1.8) may be given in terms of constrained integrals whose equations of motion are exactly those of dynamics in $\boldsymbol{R}^4$. The constraints will be shown to be the spin-reducing ones (1.1) and (1.4) or their generalisations and the extra dimensions are to be interpreted as those associated with central charges. We will leave to later papers analysis of the expressions (1.6), (1.7).

We proceed in § 2 to outline briefly how a constrained integral over a given space can naturally lead to dynamics on a lower-dimensional submanifold, taking as a particular case a single real free scalar field.

This analysis is extended to the case of a free spinor field. Both cases use the constraint (1.1). In § 3 we extend this analysis to more than one central charge, where (1.1) must be refined in order to have finite multiplets. We then analyse the superspace versions of these cases in §§ 4 and 5, first for $N=2$ and then for $N=4$ and 8. We give a discussion of our results in § 6.

2. Free fields with one extra dimension

We will start our detailed analysis by considering a single free scalar field A in five dimensions, which we denote by (x^μ, x^5) with $1\leqslant\mu\leqslant 4$. We take x^5 to be space-like, and require the spin-reducing constraint (1.1), which in this instance becomes

$$(\Box-\partial_5^2)A(x,x^5)=0. \tag{2.1}$$

Condition (2.1), regarded as a second-order differential equation in the independent variable x^5, will have solutions determined by the boundary values

$$A(x,x_0^5)=A_0(x), \qquad \partial_5 A(x,x_0^5)=A_1(x) \tag{2.2}$$

where $x^5 = x_0^5$ is the boundary of the central charge region. We wish to construct a Lagrangian in the full five-dimensional space of (x^μ, x^5) for the field $A(x, x^5)$ with the constraint (2.1) so that the resulting field equations are those corresponding to the consideration of ∂_5 as an off-shell central charge in four dimensions. We thus desire the field equations

$$\Box A_0(x) = 0, \qquad A_1(x) = 0. \tag{2.3}$$

We can see how to achieve these field equations by a naive argument which we will make precise later.

If we take $A(x, x^5)$ to have length dimension -1 then we expect our action to be of quadratic form

$$\int d^4x \, dx^5 \, A(x, x^5) T A(x, x^5) \tag{2.4}$$

where T has therefore to be a differential operator of dimension -3. This may be formed from $\Box\partial_5$ or ∂_5^3, which are equal by (2.1). We thus take the action

$$\int d^4x \, dx^5 \, A(\Box + \partial_5^2)\partial_5 A. \tag{2.5}$$

We may rewrite (2.5), by integration by parts, as

$$-\tfrac{1}{2}\int d^4x \, dx^5 \, \partial_5[(\partial_\mu A)^2 - (\partial_5 A)^2]. \tag{2.6}$$

We might suppose that (2.6) gives zero, since it is the integral of a total derivative, and it is usual to drop contributions from infinity. However, if we notice that the term being differentiated in (2.6) is the four-dimensional Lagrangian density needed to produce the equations of motion (2.3), we will proceed more cautiously. If we interpret the integral over x^5 in (2.4) as restricted to the half-space $x^5 \geqslant x_0^5$ and neglect the contribution in (2.6) from $x^5 \to +\infty$ we immediately obtain the action

$$\tfrac{1}{2}\int d^4x[(\partial_\mu A_0)^2 - A_1^2]. \tag{2.7}$$

This action clearly gives the field equations (2.3) on variation in A_0 and A_1, now without the constraint (2.1).

Of course the derivation of (2.7) was ambiguous since if the constraint (2.1) were used (2.5) could be re-expressed solely in terms of the first or second terms in (2.6), thus giving only the first or second terms in (2.7). What we need is to consider variation of the action of the form (2.5) more carefully when we take account of the constraint (2.1). Since this is a differential constraint we must use variational methods which are sensitive to such a feature.

We thus have to consider the problem of determining the equations satisfied by an extremum A of an action functional $I(A)$ where A is subject to the further constraint $h(A) = 0$. This is set in the general context of an optimisation problem in the presence of an equality constraint, for which a precise mathematical theory is known (Pontryagin *et al* 1962, Luenberger 1969, Lions 1971). The particular result of relevance is the Lagrange multiplier theorem, well known in mathematical physics, whose precise form (which is the basis of our work) is as follows.

Theorem (*Lagrange Multiplier*). (See Pontyagin *et al* (1982), Luenberger (1969) and Lions (1971).) If a suitably differentiable real valued functional $I(A)$ on a Banach space X has an extremum at the regular point a under the constraint $h(A)=0$ (where h maps X into another Banach space Y), then there exists an element $y^*\in Y^*$ so that the Lagrange functional $I(A)+y^*h(A)$ is stationary at a, or

$$I'(a)+y^*h'(a)=0. \tag{2.8}$$

The equation (2.8) is the expected Euler–Lagrange equation with Lagrange multiplier y^*. We note that A must belong to a vector space, so that if the constraint $h(A)$ also involves specification of boundary values such as in (2.2) then these must be subtracted from A in a manner dependent on their specific nature, as we will see shortly in specific cases. As a simple example suppose we were considering the space of all real continuous functions $x(t)$ of one variable t on the interval $[a, b]$ such that $x(a)=0$, $x(b)=c$. Such functions do not form a vector space. The problem can be re-expressed in terms of the continuous functions $y(t)=x(t)-c(t-a)/(b-a)$, which do vanish at both end-points and so form a vector space. We have not spelt out the exact nature of the differentiability required on I and h nor other functional analytic niceties since we wish to concentrate on the main principles. However, we should remark that one of the important properties is that a be a regular point of the constraint h (Pontryagin *et al* 1962, Luenberger 1969, Lions 1971). That is to say the Fréchet derivative $h'(a)$ maps X onto Y. This assumption is crucially related to the generalisation of the classical inverse function theorem.

Let us use the theorem to discuss the real scalar free field satisfying (2.1) and (2.2). We take for the initial form of the action the expression (2.4) with $T=\Box\partial_5$. Let us define the two-component vector u with $u^T=(A, \partial_5 A)$. We may express the functions $u(x, x^5)$

$$u^T=(u_1, u_2)=(v_1(x, x^5)+A_1(x), v_2(x, x^5)+\alpha(x^5)A_2(x)),$$
$$v_1(x, 0)=v_2(x, 0)=0,$$

where $\alpha(x^5)$ is a fixed sufficiently differentiable function with $\alpha(0)=1$ and $\alpha(\infty)=0$. We note that we may avoid problems of convergence of integrals at ∞ by choosing the range of x^5 to be a finite interval $[a, b]$; all our results will be valid in that case also, so will not be given in detail.

The weakest assumption we need to make on our functions is that on-shell the condition $v_2(x, \infty)=0$ is satisfied. This is to prevent the appearance of a propagating massless scalar mode with non-trivial central charge, so destroying the off-shell character of the central charge.

We note that the boundary condition that we are imposing on $v_2(x, x^5)$ may not be the most appropriate for quantisation, since it is only an on-shell restriction. The more general condition $\partial_5 u_1(x, \infty)=0$ off-shell, i.e. without requiring $\Box A=0$, may then be more appropriate, but since it gives the same field equation as the on-shell constraint (being stronger), we will only use the on-shell constraint here.

Let us call X the function space of two-component vectors u with this boundary condition and Y the function space of two component vectors with any boundary condition. The constraint (2.1) can be rewritten as

$$\partial_5 u=Mu, \qquad M=\begin{pmatrix}0 & 1\\ \Box & 0\end{pmatrix}, \tag{2.9}$$

and the action as

$$I=\int \mathrm{d}^4x\ \mathrm{d}x^5\ u^TNu, \qquad N=\begin{pmatrix}0 & \Box\\ \Box & 0\end{pmatrix}. \tag{2.10}$$

Let T be the map $T\equiv(\partial_5-M):X\to Y$. The Fréchet differential of T is $\delta T(u)=T'u=\partial_5u-Mu$. Given $y\in Y$, the fundamental theorem of differential equations assures us of the existence of u such that $T'u=\partial_5u-Mu=y$, which shows that T' maps X onto Y, therefore the regularity condition is satisfied. Without loss of generality we can express the problem (2.9), (2.10) in the following canonical form, which resembles the usual optimal control problem (Pontryagin *et al* 1962, Luenberger 1969, Lions 1971)

$$I=\int \mathrm{d}^4x\ \mathrm{d}x^5(v+u_0)^TN(v+u_0), \qquad \partial_5v=Mv+Mu_0, \tag{2.11}$$

where $u=v+u_0$, $v|_{x^5=x^5_0}=0$; u_0 is the initial data associated with the differential constraint.

This problem can be generalised in the following way:

$$I=\int \mathrm{d}^4x\ \mathrm{d}x^5\ \mathscr{L}(v,u_0),$$

for v satisfying

$$\partial_5v=f(v,u_0), \qquad v|_{x^5=x^5_0}=0. \tag{2.12}$$

We are now able to analyse the field equations and boundary value conditions associated with this general problem.

The Lagrange theorem and Riesz representation (Pontryagin *et al* 1962, Luenberger 1969, Lions 1971) theorem yield the existence of λ such that

$$\int \mathrm{d}^4x\ \mathrm{d}x^5\frac{\delta\mathscr{L}}{\delta v}\cdot\delta v+\int \mathrm{d}^4x\ \mathrm{d}\lambda\left(\delta v-\int_{x^5_0}^{x^5}\mathrm{d}x^5\frac{\delta f}{\delta v}\cdot\delta v\right)=0,$$

$$\int \mathrm{d}^4x\ \mathrm{d}x^5\frac{\delta\mathscr{L}}{\delta u_0}\cdot\delta u_0-\int \mathrm{d}^4x\ \mathrm{d}\lambda\int_{x^5_0}^{x^5}\mathrm{d}x^5\frac{\delta f}{\delta u_0}\cdot\delta u_0=0.$$

Without loss of generality we may take $\lambda|_{x^5=\infty}=0$ (since only the differential $\mathrm{d}\lambda$ appears in the above), and integration by parts gives

$$\int \mathrm{d}^4x\ \mathrm{d}x^5\left(\frac{\delta\mathscr{L}}{\delta v}\cdot\delta v+\frac{\delta f}{\delta v}\cdot\lambda\cdot\delta v\right)+\int \mathrm{d}^4x\ \mathrm{d}\lambda\cdot\delta v=0, \tag{2.13a}$$

$$\int \mathrm{d}^4x\ \mathrm{d}x^5\left(\frac{\delta\mathscr{L}}{\delta u_0}\cdot\delta u_0+\frac{\delta f}{\delta u_0}\cdot\lambda\delta u_0\right)=0. \tag{2.13b}$$

(2.13a) holds for all δv, in particular for δv vanishing at $x^5=x^5_0$. Integration by parts in (2.13a) gives

$$\int \mathrm{d}^4x\ \mathrm{d}x^5\left(\frac{\delta\mathscr{L}}{\delta v}\cdot\delta v+\frac{\delta f}{\delta v}\cdot\lambda\cdot\delta v-\lambda\,\partial_5\delta v\right)=0. \tag{2.14}$$

It is a well known result that (2.14) implies differentiability of λ (Pontryagin *et al* 1962), therefore we have

$$\int d^4x \, dx^5\left(\frac{\delta \mathscr{L}}{\delta v}+\frac{\delta f}{\delta v}\cdot\lambda+\partial_5\lambda\right)\delta v=0,$$

which implies

$$\delta\mathscr{L}/\delta v+(\delta f/\delta v)\cdot\lambda+\partial_5\lambda=0. \tag{2.15a}$$

(2.13*b*) yields

$$\int dx^5\left(\frac{\delta \mathscr{L}}{\delta u_0}+\frac{\delta f}{\delta u_0}\cdot\lambda\right)=0. \tag{2.15b}$$

We also have as a field equation the differential constraint

$$\partial_5 v=f(v,u_0). \tag{2.15c}$$

(2.15) with the boundary conditions $v|_{x^5=x^5_0}=0$, $\lambda|_{x^5=\infty}=0$ are the field equations for the general problem (2.12). We remark that we may take $\lambda|_{x^5=\infty}=0$ without loss of generality; moreover this is the unique boundary condition which implies the field equation (2.15*a*). Furthermore, in the first-order formulation we are following we have

$$\delta\mathscr{L}/\delta v=\delta\mathscr{L}/\delta u_0, \qquad \delta f/\delta v=\delta f/\delta u_0,$$

which are a direct consequence of the decomposition $u=v+u_0$. Therefore from (2.15*b*) we obtain

$$\lambda|_{x^5=x^5_0}=\lambda|_{x^5=\infty}=0. \tag{2.15d}$$

Finally we may formulate an unconstrained action for our problem (2.12)

$$I=\int d^4x\, dx^5[\mathscr{L}(v,u_0)+\lambda^T(f(v,u_0)-\partial_5 v)], \tag{2.16}$$

with the boundary conditions $\lambda|_{x^5=\infty}=0$, $v|_{x^5=x^5_0}=0$. These mixed boundary value conditions are a general property of the cone formulation we are following. This is in fact the usual mixed boundary value conditions one has in the general optimal control problem (Pontryagin *et al* 1962, Luenberger 1969, Lions 1971). We are now able to apply these results to the scalar free field; we have

$$\mathscr{L}(v,u_0)=(v+u_0)^T N(v+u_0), \qquad f(v,u_0)=Mv+Mu_0.$$

We directly get the field equations

$$2N(v+u_0)+M^T\lambda+\partial_5\lambda=0, \qquad Mv+Mu_0-\partial_5 v=0, \tag{2.17a, b}$$

from which we obtain the following differential equation for λ:

$$\partial_5^2\lambda+M^T\partial_5\lambda-NMN^{-1}(M^T\lambda+\partial_5\lambda)=0.$$

We also notice that $NMN^{-1}=M^T$, therefore

$$\partial_5^2\lambda=M^TM^T\lambda=\Box\lambda. \tag{2.18}$$

The boundary conditions (2.15*d*) together with this second-order differential equation yield $\lambda(x,x^5)=0$. From (2.17) we have $Nu=0$, $\partial_5 u=Mu$. The corresponding component equations are

$$\Box u_1 = \Box u_2 = 0, \qquad \partial_5 u_1 = u_2, \qquad \partial_5 u_2 = \Box u_1 = 0,$$

and from our asymptotic assumption on the function space X: $u_2|_\infty = 0$, we have

$$\partial_5 A(x, x^5) = 0, \qquad \Box A(x, x^5) = 0. \tag{2.19a, b}$$

We have thus obtained the correct field equations, (2.19), in four-dimensional space–time from the five-dimensional action (2.10). In the process we have discovered that the x^5 integration is to be interpreted as over a half-line to $+\infty$ (or $-\infty$), and that the constrained action

$$\int d^4x \, dx^5 \, A\Box\partial_5 A \tag{2.20}$$

is to be interpreted as a half-space integral. The other aspect of the constrained variational problem is that we have now to take account of variations with respect to the boundary values as well as with respect to fields v and λ satisfying homogeneous boundary value conditions. The conclusion of our analysis is that the dynamics for such a field can be regarded as occurring in a five-dimensional space–time in whose half-space boundary is our four-dimensional world.

We may extend the above analysis to a free chiral spinor ψ satisfying the same constraint (2.1). We may again introduce the two-component vector $u^T = (\psi, \phi)$ with $\partial_5\psi = \not\partial\phi$, and consider the constrained Lagrangian

$$\int d^4x \, dx^5 \, \bar{u}^T N u, \qquad \partial_5 u = Mu, \tag{2.21a, b}$$

where

$$N = \begin{pmatrix} 0 & \Box \\ \Box & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & \not\partial \\ \not\partial & 0 \end{pmatrix}.$$

We may consider the same behaviour of u at infinity as before: $\partial_5 u = 0$ at $x^5 = \infty$. The variational equations are as before, and we have the field equations

$$\Box u = 0, \tag{2.22}$$

which together with the constraint (2.21*b*) yield $\partial_5^2 u = M^2 u = \Box u = 0$. Therefore, using the boundary condition, we have $\partial_5 u = 0$, or equivalently,

$$\partial_5\psi(x, x^5) = 0, \qquad \partial_5\phi(x, x^5) = 0, \tag{2.23}$$

and from (2.21*b*)

$$\not\partial\psi(x, x^5) = 0, \qquad \not\partial\phi(x, x^5) = 0. \tag{2.24}$$

Equations (2.24) are the x-space equations appropriate for massless spinors with an off-shell central charge (which vanishes on-shell).

Similar actions may easily be given for higher spin fields by the addition of suitable Lorentz indices to the scalars and spinors already discussed. Mass may also be included by the modification $\not\partial \to (\not\partial + im)$, $\Box \to (\Box + m^2)$ in the above expressions. We note that m corresponds to a mass in four dimensions, there are no massive excitations in the x^5 direction which have to be removed to unobservably large energies by spontaneous compactification or other techniques (Witten 1981, Salam and Strathdee 1981).

We finally add that even for an unconstrained field in four dimensions we may define a constrained action in five dimensions along similar lines. We now consider the five-dimensional action (2.20), but now with the constraint

$$A(x, \infty)=0. \tag{2.25}$$

Since the action (2.20) is a total derivative, and so only with dependence on $A(x, x_0^5)$ and $A(x, \infty)$, the constraint (2.25) will therefore lead to the field equation (2.19*b*) at $x^5 = x_0^5$. We note that (2.25) cannot be used with constraint (2.1) since the wrong field equations then result. We can expect to be able to write down the same five-dimensional action for either of the constraints (2.1) or (2.25). This will be of importance if we wish to use a unified formulation for multiplets with or without degenerate central charges, as will be needed for a superfield formulation of the solution of Taylor (1981), which involve the former as compensating multiplets for the latter.

3. Free fields with two extra dimensions

We gave in the introduction good reasons for trying to integrate over the extra central charge dimensions to allow us to construct extended N-SYMs or N-SGRs, beyond the $N = 3$ barrier, in terms of maximally extended superfields. Even for $N = 2$ this was shown to require the use of at least two extra dimensions. We now extend the discussion of the last section to this case, with the additional variable being x^6. The constraint (2.1) now becomes

$$(\Box-\partial_5^2-\partial_6^2)A(x, x^5, x^6) = 0. \tag{3.1}$$

If we wish to reduce the dynamics to four-dimensional free field equations for a set of fields at $x_5 = x_6 = 0$ (we take $x_0^5 = 0$ and $x_0^6 = 0$ for simplicity here, though these should in general be regarded as arbitrary points) we find that in general this set of fields will be infinite in number. One way to see this is in terms of the independent derivatives which we can construct from powers of ∂_5 and ∂_6 using the constraint (3.1); these consist of the sequence of powers $\boldsymbol{\partial} = \{\partial_5, \partial_5^{n-1}\partial_6\}_{n\geq 1}$. In other words there will be an infinite sequence of boundary value fields composed of $A(x, 0)$ and $\boldsymbol{\partial}A(x, 0)$. Dividing by suitable inverse powers of $\Box$ to obtain correct canonical dimensions we expect that half of these will propagate, half disappear, in four dimensions (Gorse *et al* 1983).

To obtain a finite number of boundary value fields we can reduce ∂ to a finite set by various further constraints beyond (3.1). The most powerful is that for some real constant b

$$\partial_6 A = b\,\partial_5 A. \tag{3.2}$$

Constraint (3.2) arises in the decomposition of the general degenerate representation for $N = 2$ SUSY into irreducible representations (Restuccia and Taylor 1983). A more general constraint is that for some constant c,

$$\partial_5^2 A = c\Box A, \qquad \partial_6^2 A = (1-c)\Box A. \tag{3.3}$$

This constraint arises naturally in the decomposition of degenerate $N = 4$ SUSY representations with two real independent central charges into irreducible ones (Bufton and Taylor 1983a).

We first discuss (3.2). On combination with (3.1) we have (3.3) with $c=(1+b^2)^{-1}$. We may thus use the method of § 2; we introduce the four-component vector $u^T=(A, \partial_5 A, \partial_6 A, \partial_5\partial_6 A)$ with the same asymptotical behaviour as before: $\partial_5 A = \partial_6 A = \partial_5\partial_6 A = 0$ at $x^5 = x^6 = \infty$. Then we can rewrite the constraints (3.1) and (3.2) as

$$\partial_5 u = Mu, \qquad \partial_6 u = bMu, \tag{3.4a, b}$$

where

$$M=\begin{pmatrix} 0 & 1 & 0 & 0\\ c\Box & 0 & 0 & 0\\ bc\Box & 0 & 0 & 0\\ 0 & 0 & c\Box & 0\end{pmatrix}.$$

In order to formulate an unconstrained action with independent Lagrange multipliers we decompose u in the following way,

$$u = v + w + u_0. \tag{3.5}$$

where

$$v = v(x, x^5, x^6), \qquad v|_{x^5=0} = 0, \tag{3.6a}$$

$$w = w(x, x^6), \qquad w|_{x^6=0} = 0. \tag{3.6b}$$

u_0 as usual is the independent initial data of the system (3.4). In terms of (3.5) the constraints (3.4) are equivalent to

$$\partial_5 v = Mv + Mw + Mu_0, \qquad \partial_6 w = bMw + bMu_0. \tag{3.7a, b}$$

Given u_0, (3.7*b*) gives the evolution on the hypersurface x^6 = constant, while (3.7*a*) propagates the information from x^6 = constant to the whole x^5, x^6 space.

We take the unconstrained action with Lagrange multipliers to be

$$I=\int d^4x\, dx^5\, dx^6\{(v+w+u_0)^T N(v+w+u_0)+\lambda_5^T[M(v+w+u_0)-\partial_5 v]\}$$
$$+\int d^4x\, dx^6\, \lambda_6^T(bMw+bMu_0-\partial_6 w), \tag{3.8}$$

where

$$N=\begin{pmatrix} 0 & & & \Box\\ & & \Box & \\ & \Box & & 0\\ \Box & & & \end{pmatrix}.$$

We can impose without loss of generality the boundary value conditions

$$\lambda_5|_{x^5=\infty}=0, \qquad \lambda_6|_{x^6=\infty}=0, \tag{3.9a, b}$$

and we can prove in the same way as before that these are the required conditions to assure a satisfactory continuity behaviour of λ_5 and λ_6.

The field equations are, together with (3.7),

$$2Nu + M^T\lambda_5 + \partial_5\lambda_5 = 0, \tag{3.10a}$$

$$\int dx^5(2Nu + M^T\lambda_5) + bM^T\lambda_6 + \partial_6\lambda_6 = 0, \tag{3.10b}$$

$$\int dx^6\, dx^5(2Nu + M^T\lambda_5) + \int dx^6\, bM^T\lambda_6 = 0. \tag{3.10c}$$

In the usual way we get from (3.10c)

$$\lambda_6|_{x^6=0} = \lambda_6|_{x^6=\infty} = 0, \tag{3.11a}$$

and from (3.10b) we obtain

$$\lambda_5|_{x^5=0} + bM^T\lambda_6 + \partial_6\lambda_6 = 0. \tag{3.11b}$$

(3.11) implies as before $\lambda_6(x, x^6) = 0$. We thus get from (3.11b)

$$\lambda_5|_{x^5=0} = 0, \tag{3.12}$$

which together with (3.9a), (3.10a) and (3.7) imply $\lambda_5(x, x^5, x^6) = 0$. Consequently, we have from (3.10a)

$$Nu = 0, \tag{3.13}$$

or equivalently

$$\Box u = 0. \tag{3.14}$$

We also have from the constraint (3.7)

$$\partial_5^2 A = c\Box A = 0, \qquad \partial_6^2 A = b^2 c\Box A = 0. \tag{3.15a, b}$$

Using now the boundary conditions on A at $x^5 = x^6 = \infty$ we get

$$\partial_5 A(x, x^5, x^6) = \partial_6 A(x, x^5, x^6) = 0, \qquad \Box A(x, x^5, x^6) = 0. \tag{3.16}$$

When (3.3) alone is used, we may consider the four-component vector u which satisfies the constraints

$$\partial_5 u = M_5 u, \tag{3.17}$$

$$\partial_6 u = M_6 u, \tag{3.18}$$

where

$$M_5 = \begin{pmatrix} 0 & 1 & & \\ c\Box & 0 & \multicolumn{2}{c}{0} \\ & & 0 & \Box \\ \multicolumn{2}{c}{0} & c & 0 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & \Box \\ (1-c)\Box & 0 & & \\ 0 & (1-c) & \multicolumn{2}{c}{0} \end{pmatrix}.$$

The full dimensional action with Lagrange multipliers is now

$$I = \int d^4x\, dx^5\, dx^6 [u^T N u + \lambda_5^T (M_5 - \partial_5) u] + \int d^4x\, dx^6\, \lambda_6^T (M_6 - \partial_6)(w + u_0), \tag{3.19}$$

where we are using the same decomposition (3.5) for u, and

$$N = \begin{pmatrix} & & & \Box^2 \\ & & \Box & \\ & \Box & & \\ \Box^2 & & & \end{pmatrix}.$$

The equations of motion are so similar to the ones discussed already that we do not discuss them in detail but just state that four-dimensional equations are

$$u_2 = u_3 = 0, \qquad \partial_5 u_1 = \partial_6 u_1 = \partial_5 u_4 = \partial_6 u_4 = 0, \qquad \Box u_1 = \Box u_4 = 0.$$

There are thus two propagating massless scalars, u_1 and u_4, and two vanishing auxiliary fields u_2, u_3. This agrees with the component content of the two-central charge multiplet of $N=4$ SUSY (Bufton and Taylor 1983a).

It is straightforward to extend the above analysis to the case of three or more extra dimensions. In the case of three extra dimensions, say x^5, x^6, x^7, the spin-reducing constraint (3.1) now becomes

$$(\Box-\partial_5^2-\partial_6^2-\partial_7^2)A(x, x^5, x^6, x^7)=0. \tag{3.20}$$

We may again deal with infinite multiplets defined by the boundary conditions at the edge of the eighth-space $x^5 \geqslant 0$, $x^6 \geqslant 0$, $x^7 \geqslant 0$ by introducing a sequence of differential operators extending $\boldsymbol{\partial}$ defined earlier but now including ∂_7 and taking account of (3.20) instead of (3.1).

To avoid the expected infinite set of propagating fields we may impose conditions similar to those of (3.2) or (3.3). For $N=4$ SUSY the first of these constraints corresponds to complexifying the central charges, the second to including further central charges in independent directions. Actions can be constructed by analogy to those given above and shown to give satisfactory equations of motion. Similar extension to include higher spin fields is also possible.

We may finally also extend the analysis of an x^5-independent scalar to the case of an x^5- and x^6-independent one. This is identical to the discussion associated with (2.25); there is only the non-trivial equation of motion (2.19*b*) at $x^5=0$.

4. Interactions in extra dimensions

We have so far only discussed free field examples of how we may embed our four-dimensional world as the 'edge' of a half-space (for one extra dimension) or of an $(n+1)$th-space for n extra dimensions. Even in those cases we did not consider Abelian gauge fields in any detail. We will attempt to remedy that omission, and build interacting theories by means of the minimal gauge principle. It appears more difficult to develop our theory to include arbitrary self-interactions, a situation possibly to be welcomed rather than otherwise; we will return to this in our final discussion.

We restrict ourselves to one extra dimension, though extension to larger numbers of dimensions can be given straightforwardly by the methods we have discussed so far; again we take the half-space to be $0 \leqslant x^5 < \infty$, though the lower limit is to be regarded as an arbitrary value. We will also only consider in detail a charged scalar field A in minimal interaction with a U(1)-gauge field A_μ, so we wish to obtain at least the four-dimensional equations of motion

$$D_\mu D^\mu A=0, \qquad D_\mu=\partial_\mu-\mathrm{i}A_\mu, \tag{4.1}$$

$$\partial^\nu F_{\nu\mu}(A_\lambda)=-J_\mu, \qquad J_\mu=\mathrm{i}A^*\overleftrightarrow{\partial}_\mu A. \tag{4.2), (4.3}$$

There are four cases to consider, according as whether the matter or the gauge field has no dependence on x^5 or satisfies an interacting analogue of the spin-reducing constraint (2.1). We take first the case

$$\partial_5 A=\partial_5 A_\mu=0. \tag{4.4}$$

We discussed the non-interacting version of this case at the end of § 2, as far as the scalar field A was concerned. Let us analyse the vector field A_μ in the absence of the scalar. As usual we introduce the two-component vector u_μ with

$u_\mu^T = (A_\mu, \partial_5 A_\mu)$. The simplest five-dimensional action to introduce has Lagrange density $u_\mu^T N^{\mu\nu} u_\nu$, where

$$N_{\mu\nu} = \begin{pmatrix} 0 & \bar{\eta}_{\mu\nu}\Box \\ \bar{\eta}_{\mu\nu}\Box & 0 \end{pmatrix},$$

with $\bar{\eta}_{\mu\nu} = \delta_{\mu\nu} - \partial_\mu \partial_\nu / \Box$. The constraint is $\partial_5 u_\mu = 0$, and by means of the Lagrange multiplier theorem the unconstrained Lagrangian is

$$\int d^4x\, dx^5 (u_\mu^T N^{\mu\nu} u_\nu + \lambda^{T\mu} \partial_5 u_\mu) \tag{4.5}$$

where the integral is taken as usual over the half-space $x^5 \geqslant 0$. The field equations are, as before,

$$2N_{\mu\nu} u^\nu - \partial_5 \lambda_\mu = 0, \tag{4.6}$$

$$\int dx^5 [2N_{\mu\nu} u^\nu] = 0, \qquad \partial_5 u_\mu = 0. \tag{4.7), (4.8}$$

Combining (4.6) and (4.7) gives $\lambda_\mu(x, 0) = 0$, and from (4.6) and (4.8), $\partial_5^2 \lambda_\mu = 0$. We therefore have the expected field equations

$$\partial^\nu F_{\mu\nu}(A_\lambda) = \partial_5 A_\mu = 0 \tag{4.9}$$

where $F_{\mu\nu}(A_\lambda) = \partial_{[\mu} A_{\nu]}$.

In the presence of the charged scalar field A also satisfying (4.4) we introduce the further complex two-component scalar u with $u^T = (A, \partial_5 A)$. It is appropriate to introduce the current two-component vector $J_\mu(u)$, with

$$J_{\mu i}(u) = u^+ L_i \overleftrightarrow{\partial}_\mu u \qquad (i = 1, 2), \tag{4.10}$$

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we add to the Lagrange density $u_\mu^T N^{\mu\nu} u_\nu$ for the free vector field, the expressions $u^+ Nu$ for the free scalar field, with $N = \left(\begin{smallmatrix} 0 & \Box \\ \Box & 0 \end{smallmatrix}\right)$, and $\mathrm{i} u_\mu^T \sigma_1 J^\mu(u) + u_\mu^T L_1 u_\mu u^+ L_2 u + u_\mu^T L_2 u_\mu u^+ L_1 u$ for the interaction term. The total unconstrained five-dimensional Lagrangian is therefore

$$\begin{aligned}(\tfrac{1}{2} u_\mu^T N^{\mu\nu} u_\nu + u^+ Nu + \mathrm{i} u_\mu^T J^\mu(u) + u_\mu^T L_1 u^\mu u^+ L_2 u \\ + u_\mu^T L_2 u^\mu u^+ L_1 u + \lambda^{T\mu} \partial_5 u_\mu + \lambda^+ \partial_5 u + \lambda \partial_5 u^+).\end{aligned} \tag{4.11}$$

The field equations are therefore

$$N_{\mu\nu} u^\nu + \mathrm{i} J_\mu(u) + 2L_1 u^\mu (u^+ L_2 u) + 2L_2 u^\mu (u^+ L_1 u) - \partial_5 \lambda^\mu = 0, \tag{4.12}$$

$$Nu - u_\mu^T (\delta J^\mu(u)/\delta u^+) - \partial_5 \lambda + (u_\mu^T L_1 u^\mu) L_2 u + (u_\mu^T L_2 u^\mu) L_1 u = 0, \tag{4.13}$$

$$\int_0^\infty dx^5 [N_{\mu\nu} u^\nu + \mathrm{i} J_\mu(u) + 2L_1 u^\mu (u^+ L_2 u) + 2L_2 u^\mu (u^+ L_1 u)] = 0, \tag{4.14}$$

$$\int_0^\infty dx^5 [Nu - u_\mu^T (\delta J^\mu(u)/\delta u^+) + (u_\mu^T L_1 u^\mu) L_2 u + (u_\mu^T L_2 u^\mu) L_1 u] = 0, \tag{4.15}$$

$$\partial_5 u_\mu = \partial_5 u = 0. \tag{4.16}$$

From (4.12) and (4.14) together with the boundary condition $\lambda_\mu(x,\infty)=0$ we obtain $\lambda_\mu(x,0)=0$, and similarly from (4.13) and (4.15) we have $\lambda(x,0)=0$. Moreover the homogeneous constraints (4.16) imply $\partial_5^2\lambda_\mu=\partial_5^2\lambda=0$, so that $\lambda_\mu(x,x^5)=\lambda(x,x^5)\equiv 0$. We therefore obtain the four-dimensional field equations (4.1)–(4.3) for the boundary values $A(x,0)$ and $A_\mu(x,0)$. The above construction (4.11) had to be successful since it is the action of ∂_5 in the correct four-dimensional action to reproducing (4.1)–(4.3). We note that the extension to the non-Abelian case is most easily achieved by defining the field strength two-component vector $U_{\mu\nu}(u_\lambda)$ defined by

$$U^a_{\mu\nu i}(u_\lambda)=\partial_{[\mu}u^a_{\nu]i}+\mathrm{i}f^{abc}u^T_{[\mu b}L_iu_{\nu]c} \tag{4.17}$$

where a denotes the adjoint representation of the gauge group G with structure constants f^{abc} and L_i $(i=1,2)$ are given in (4.10). The five-dimensional Lagrangian (4.11) is now to be modified by the replacement of the first term by

$$\tfrac{1}{2}U^{Ta}_{\mu\nu}U^{\mu\nu a} \tag{4.18}$$

with obvious modification of the current $-u_\mu$ interaction term and others in (4.11) to take account of the representation labels also present on the scalar field u. The remaining argument to obtain the usual four-dimensional field equations is little changed, so we do not give it here.

We next turn to the case when the scalar field A carries an off-shell central charge, so satisfies (2.1), whilst A_μ still satisfies (4.4). The only change to (4.11) is the replacement of ∂_5 in the last two terms in (4.11) by $(\partial_5 M)$, with M given by (2.9), though with $\Box$ replaced by $D^*_\mu D^\mu$, $D_\mu=\partial_\mu-\mathrm{i}(1,0)^Tu_\mu$. Similar replacement must be made in (4.13) (in this case $\partial_5\to\partial_5+M^T$) and the condition on u in (4.16). The argument proceeds as before to show $\lambda^\mu(x,x^5)\equiv 0$, but the discussion of $\lambda(x,x^5)$ needs a little more care. As usual the boundary conditions and (4.13), (4.15) may be used to show $\lambda(x,0)=0$. We now wish to combine (4.13) and the new constraint (2.11) on u to obtain a second-order differential equation in ∂_5 for λ.

The detailed form of (4.13) is

$$[N-u^T_\mu L\vec{\partial}_\mu+2(u^T_\mu L_1u^\mu)L_2+2(u^T_\mu L_2u^\mu)L_1]u=(\partial_5+M^T)\lambda. \tag{4.19}$$

If we can invert the differential operator on the LHS then the resulting equation

$$(\partial_5-M)(N-u^T_\mu L\vec{\partial}_\mu+(u^T_\mu L_1u^\mu)L_2+(u^T_\mu L_2u^\mu)L_1)^{-1}(\partial_5+M^T)\lambda=0 \tag{4.20}$$

has the required form. On use of the constraint (4.4) for A_μ we find that the LHS of (4.19) is simply $(N-A^\mu\vec{\partial}_\mu+2A^\mu A_\mu)$, whose inverse will exist for a large class of A_μ. Under that assumption the argument leading to the four-dimensional field equation (4.1) for $A(x,0)$ and $\partial_5A(x,0)=0$ will go through as before.

The problem of constructing a satisfactory five-dimensional theory for which the gauge field has a non-trivial central charge transformation is not as simple due to the presence of the constraint and of the nonlinear structure introduced by the current $J_\mu(u)$. When $J_\mu=0$ we may take the constraint on A_μ as in (2.1), and thus have the Lagrangian (4.5) with ∂_5 in the last term in (4.5) replaced by (∂_5-M). We also need to constrain the longitudinal part of A_μ by the condition ∂

$$\partial_5(\partial_\mu A^\mu)=0. \tag{4.21}$$

We note that (4.21) is gauge invariant under the gauge transformation $\delta A_\mu=\partial_\mu\Lambda(x)$, with x^5-independence of Λ. We must add to the Lagrangian (4.5) a further Lagrange multiplier term ensuring (4.21), of form

$$\int d^5x\,\mu\cdot\partial_5(\partial_\mu A^\mu)$$

$$2N_{\mu\nu}u^\nu-(\partial_5+M^T)\lambda^T_\mu=0, \tag{4.22}$$

$$\int_0^\infty dx^5(2N_{\mu\nu}u^\nu-M^T\lambda_\mu)=0, \tag{4.23}$$

$$(\partial_5-M)u_\mu=0, \tag{4.24}$$

$$u_\mu=\tfrac{1}{2}N^{-1}_{\mu\nu}(\partial_5+M^T)\lambda^\nu+\partial_\mu u \tag{4.25}$$

where $N^{-1}_{\mu\nu}=\sigma_1\Box^{-1}\eta_{\mu\nu}$ with σ, the usual Pauli matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$,

$$[(\partial_5+M^T)\lambda^L_\mu]_2=\partial_\mu\mu \tag{4.26}$$

and the constraint (4.21). If we apply the operator $(\partial_5-M)\sigma_1$ to (4.21) and use (4.25) we obtain $\frac{1}{2}(\partial_5-M)\sigma_1(\partial_5+M^T)\lambda_\mu=0$. This is second order in ∂_5, so $\lambda_\mu(x,x^5)\equiv0$. The boundary value field equations are therefore

$$\partial^\mu F_{\mu\nu}(A_\lambda)(x,0)=0. \tag{4.27}$$

We may use the spin-reducing constraint on the transverse part A^{T}_μ of A_μ and the vanishing of BA^{T}_μ to deduce the x^5-independence of $\partial_5A^{\mathrm{T}}_\mu$; the boundary constraint that $\partial_5A^{\mathrm{T}}=0$ at $x^5=\infty$ implies that

$$\partial_5A_\mu=\partial_\mu\phi \tag{4.28}$$

for some scalar ϕ. The constraint (4.21) then leads to the expected equation of motion

$$\Box\phi=0. \tag{4.29}$$

Thus $\phi(x,0)$ is a propagating massless scalar field, as expected.

We may try to extend the above in various directions. Firstly we can couple the current $J_\mu(u)$ as in (4.11). However, the presence of $J_\mu(u)$ in (4.21) prevents the use of (4.24) to deduce (4.26). Secondly we may extend to the non-Abelian case, so using (4.17) and (4.18). Due to the nonlinearity of the modified version of (4.21) we cannot solve simply as in (4.25), and here again we do not obtain the vanishing of $\lambda_\mu(x,x^5)$. Finally we may take the non-Abelian case in the presence of matter with either trivial or non-trivial central charge dependence. Both these cases add further complications to an already difficult situation, and we have not yet obtained a solution. We conclude that gauge fields are easy to fit into the central charge framework if they have trivial central charge properties, still possible to incorporate if they are Abelian, and decoupled from all matter if they have non-trivial central charge features, and very difficult, if not impossible in that case to include if they are non-Abelian or coupled to charged matter. We hope to return to a further analysis of this question elsewhere. We finally add that both matter (scalars or spinors) and gauge fields have been handled throughout this section by means of a first-order formalism, using two component vectors. Indeed the more obvious case of a 2×2 matrix notation for the gauge vectors does not seem easy to construct. We do need an identical framework in terms of which we may discuss all of these fields if we wish to require supersymmetry. We now have such a framework available, and could proceed to extend our earlier discussions to include SUSY invariance. We will find it more immediately if we use superfield techniques for this approach, and turn to that now.

5. Superfields in extra dimensions

We have found that it is possible to describe the major aspects of free and interacting field theories in four dimensions as having the same field equations as if the theory arose from a field theory described in higher dimensions with a suitable constraint. However, the constructions appear somewhat artificial, whilst the remarks in the introduction indicate that the extra 'central charge' dimensions are essential to make use of the full superspace in extended SUSY theories. We will now discuss these cases in detail.

Let us first consider an artificial use of the full superspace measure with a constraint. We take $N = 1$ SUSY and consider a chiral superfield Φ which has the constraint

$$\bar{D}_{\dot\alpha}\Phi = 0. \tag{5.1}$$

We assume Φ has canonical length dimension -1, so that the only superfield action available on using the full measure $d^4x\, d^2\theta\, d^2\bar\theta$ is

$$\int d^4x\, d^2\theta\, d^2\bar\theta\, \Phi^+\Phi. \tag{5.2}$$

We may apply the theorem of § 2, with the Lagrange multiplier being a spinor superfield $\bar\lambda^{\dot\alpha}$ and unconstrained action

$$\int d^4x\, d^2\theta\, d^2\bar\theta[\Phi^+\Phi - \bar\lambda^{\dot\alpha}\bar{D}_{\dot\alpha}\Phi + \lambda^\alpha D_\alpha\Phi^+]. \tag{5.3}$$

The resulting variational equations are (5.1) and

$$\Phi + D_\alpha\lambda^\alpha = 0. \tag{5.4}$$

From (5.4) we may immediately deduce

$$D^\alpha D_\alpha\Phi = 0 \tag{5.5}$$

which, together with (5.1), is the well known equation of motion for a chiral superfield. We note that the superspace integration does not play the same role as the x^5 or x^6 integration discussed up to now, but has its traditional form.

Let us extend our analysis to $N = 2$. The full superspace measure without central charge dimensions is $d^4x\, d^4\theta\, d^4\bar\theta$, with dimension 0. In order to consider the $N = 2$ chiral superfield Φ and the associated Lagrange density $\Phi^+\Phi$ we require at least two extra dimensions x^5 and x^6. If we are considering a central-charge independent chiral superfield then we can consider the constraints

$$\partial_5\Phi = \partial_6\Phi = 0, \tag{5.6}$$

$$\bar{D}_{\dot\alpha i}\Phi = 0 \qquad (i = 1, 2). \tag{5.7}$$

The constrained action is taken to be

$$\int d^4x\, dx^5\, dx^6\, d^4\theta\, d^4\bar\theta\, \Phi^+\Phi \tag{5.8}$$

and inclusion of Lagrange multipliers gives the unconstrained action

$$\int d^4x\, dx^5\, dx^6\, d^4\theta\, d^4\bar\theta(\Phi^+\Phi - \bar{D}_{\dot\alpha i}\bar\lambda^{\dot\alpha i}\cdot\Phi + D_{\alpha i}\lambda^{\alpha i}\Phi^+ + \lambda_5^+\partial_5\psi + \lambda_5\partial_5\psi^+)$$
$$+ \int d^4x\, dx^6\, d^4\theta\, d^4\bar\theta(\lambda_6^+\partial_6\mu + \lambda_6\partial_6\mu^+) \tag{5.9}$$

where $\Phi = \psi + \mu + \Phi_0$, $\psi|_{x^5 = x_0^5} = 0$, $\partial_5\mu = 0$, $\mu|_{x^6 = x_0^6} = 0$ and $\Phi_0 = \Phi_0(x, \theta, \bar\theta)$ is the initial data associated with the superfield Φ. This decomposition is necessary in order to satisfy the regularity condition of the constraints. The Lagrange multipliers $\bar\lambda^{\alpha i}$ and $\lambda^{\alpha i}$ are independent of x^5 and x^6 and λ_5, λ_6 satisfy the usual boundary condition. The resulting equations of motion are

$$\Phi + D_{\alpha i}\lambda^{\alpha i} - \partial_5\lambda_5 = 0, \qquad \int dx^5(\Phi + D_{\alpha i}\lambda^{\alpha i}) - \partial_6\lambda_6 = 0, \tag{5.10a, b}$$

$$\int dx^5\, dx^6(\Phi + D_{\alpha i}\lambda^{\alpha i}) = 0, \tag{5.10c}$$

$$\bar D_{\dot\alpha i}\int dx^2\, dx^6\Phi = 0, \tag{5.11}$$

$$\partial_5\psi = \partial_6\mu = 0. \tag{5.12}$$

From (5.11) and (5.12) we obtain (5.6) and (5.7). Using these results in (5.10) we get $\lambda_5 = \lambda_6 = 0$,

$$\Phi + D_{\alpha i}\lambda^{\alpha i} = 0. \tag{5.13}$$

Multiplication of (5.13) by $D^4 = \varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}D_{\alpha_1}D_{\alpha_2}D_{\alpha_3}D_{\alpha_4}$ gives the equation of motion

$$D^4\Phi = 0. \tag{5.14}$$

We note that in combination with the constraints (5.7) we obtain the χ-space equation of motion at $x^5 = x^6 = 0$

$$\Box^2\Phi = 0 \tag{5.14a}$$

which at $\theta = 0$ gives the incorrect equation of motion due to the presence of one too many powers of $\Box$ on the RHS of (5.14*a*). This is related to the component form of action arising from (5.8); in particular the contribution from the scalar $A(x, x^5, x^6) = \Phi(x, x^5, x^6, 0)$ is

$$\int d^6x\, A^+\Box^2 A. \tag{5.14b}$$

Since $\partial_5 A = \partial_6 A = 0$ then (5.14*b*) is determined purely in terms of the constant (and so boundary) value $A(x)$ of $A(x, x^5, x^6)$. Absorbing the area of (x^5, x^6) integration (assumed finite here) into $A(x)$ we thus obtain the four-dimensional action proportional to $\int d^4x\, A^+\Box^2 A$, with corresponding field equation $\Box^2 A$ identical to the $\theta = 0$ part of (5.14*a*). Imposition of the further reality constraint $D^2\Phi = \bar D^2\Phi^+$ (to make the vector field in Φ real) will still lead to (5.14*a*), as can be seen by the fact that the component form of the action still has contribution (5.14*b*) from the scalar A.

In order to obtain a satisfactory Lagrangian for this case we may proceed by following the method of § 2 more closely. Thus we introduce a set of four superfields as the components of the four-vector U with

$$U_1 = \Phi, \qquad \partial_5 U_1 = U_2, \qquad \partial_6 U_1 = U_3, \qquad \partial_5\partial_6 U_2 = U_4. \tag{5.15}$$

The superspace action (with the added reality constraint mentioned above)

$$\int_\Gamma d^6x\, d^8\theta\, U^+T\cdot\Box^{-1}U, \qquad T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.16}$$

with constraints (5.7) and (5.15) and where Γ is, as usual, the quadrant $x^5 \geqslant 0$, $x^6 \geqslant 0$, now has field equation at $x^5 = x^6 = 0$ with only one power of $\Box$ in (14*a*); the corresponding scalar action is

$$\int d^6x\, u^+ \Box u$$

where $u^+ = (A^+, \partial_5 A^+, \partial_6 A^+, \partial_5\partial_6 A^+)$. Using the constraint that $A = 0$ at $x^5 = \infty$ or $x^6 = \infty$ then we obtain the usual four-dimensional action $\int d^4x\, A^+ \Box A$.

There is a clear disadvantage of the above procedure in that the action (5.16) is explicitly non-local due to the presence of the $\Box^{-1}$ term. Whilst the approach may also be used for higher N, this non-locality increases; for $N = 4$ we must include a factor $\Box^{-3}$ in (5.16). Whilst this non-locality is absent in the component action it will be present if we try to construct self-interaction even in the case of $N = 2$. For $\Box$ must then be replaced by the square of the gauge-covariant derivative $\mathscr{D}_\mu$, whose inverse will have non-locality when expanded in powers of the gauge coupling constant. We will therefore not follow this method any further.

An alternative approach is to solve the constraint (5.7) by $\Phi = \bar{D}^4 w$ and to use the reality condition which defines the irreducible multiplet $\bar{D}^2_{ij}\Phi^+ = D^2_{ij}\Phi$. The full superspace action for Φ and w is

$$\int d^4x\, dx^5\, dx^6\, d^4\theta\, d^4\bar{\theta}\, \partial_5\partial_6(\Phi \cdot w + \bar{D}^4\lambda \cdot w + \mu w + D^2_{ij}\mu \cdot v^{ij} + \text{HC}) \tag{5.17}$$

where the Lagrange multipliers are independent of x^5 and x^6. We assume the boundary conditions $\Phi(x, \theta, \bar{\theta}, x_5, \infty) = \Phi(x, \theta, \bar{\theta}, \infty, x_6) = 0$ and w (and v^{ij}) satisfy $w(x, \theta, \bar{\theta}, x_5, \infty) = w(x, \theta, \bar{\theta}, \infty, x_6) = 0$. (5.17) gives the correct four-dimensional action $\int d^4 x\, d^4\theta (\Phi^2 + \text{HC})$.

However, we cannot use the same method for $N = 4$ or 8, since the equivalent of (5.17) cannot be given for dimensional reasons. There thus seem to be basic differences between multiplets with and without degenerate central charges when we attempt to construct full superspace actions with $N \geqslant 4$. We will have to take note of this for the construction of $N = 4$ SYM and $N = 8$ SGR (Hassoun *et al* 1983).

We now turn to the $N = 2$ hypermultiplet (Fayer 1976), which is the fundamental multiplet of the degenerate or spin reducing $N = 2$ representations. It may be described (Sohnius 1978, Taylor 1980) by the doublet superfield Φ_i with

$$D_{\alpha(i}\Phi_{j)} = \bar{D}_{\dot{\alpha}(i}\Phi_{j)} = 0. \tag{5.18}$$

It is known (Restuccia and Taylor 1983) from the representation theory of $N = 2$ spin reducing multiplets that a further reality condition must be applied in order to achieve an irreducible representation of $N = 2$ SUSY. The simplest case of this is

$$\partial_6\Phi_i = 0. \tag{5.19}$$

We take the full superspace action to be

$$\int d^4x\, dx^5\, dx^6\, d^4\theta\, d^4\bar{\theta}\, \Phi_i^+\Phi_i. \tag{5.20}$$

We take the region of integration in (x^5, x^6) to be the general quadrant $x^5 \geqslant x^5_0$, $x \geqslant x^6_0$ together with the constraints (5.18), (5.19).

We remark that the constrained action (5.20) with constraints (5.18), (5.19) is manifestly invariant under SUSY and central charge transformations. Let us prove it in detail.

Consider first central charge transformations

$$x' = x, \qquad \theta' = \theta, \qquad \bar{\theta}' = \bar{\theta}, \qquad x^{5\prime} = x^5 + a, \qquad x^{6\prime} = x^6 + b.$$

We note here that we have to transform the region of integration in the central charge dimensions, since otherwise the action (5.20) would not be invariant. We also know from the transformation property of scalar superfields

$$\Phi_i'(x, \theta, \bar{\theta}, x^5 + a, x^6 + b) = \Phi_i(x, \theta, \bar{\theta}, x^5, x^6). \tag{5.21}$$

We have

$$I' = \int \mathrm{d}\Omega \int_{x_0^{5\prime}} \int_{x_0^{6\prime}} \mathrm{d}x^{5\prime}\, \mathrm{d}x^{6\prime}\, \Phi_i^{+\prime}(x^{5\prime}, x^{6\prime}) \Phi_i(x^{5\prime}, x^{6\prime}),$$

$$I = \int \mathrm{d}\Omega \int_{x_0^5}^{\infty} \int_{x_0^6}^{\infty} \mathrm{d}x^5\, \mathrm{d}x^6\, \Phi_i^+(x^5, x^6) \Phi_i(x^5, x^6),$$

where $x_0^{5\prime} = x_0^5 + a$, $x_0^{6\prime} = x_0^6 + b$.

Consider a change of integration variable in I':

$$x^{5\prime} = x^5 + a, \qquad x^{6\prime} = x^6 + b,$$

we have

$$I' = \int \mathrm{d}\Omega \int_{x_0^5}^{\infty} \int_{x_0^6}^{\infty} \mathrm{d}x^5\, \mathrm{d}x^6\, \Phi_i^{+\prime}(x^5 + a, x^6 + b) \Phi_i'(\ldots).$$

Now using the transformation property of superfields (5.21) we obtain $I' = I$. This is the very well known result that the Lagrangian density must transform as a scalar density. This is the case because the Jacobian is 1. The same result is vaid for SUSY transformations

$$x' = x + \mathrm{i}\theta\sigma\bar{\xi} - \mathrm{i}\xi\sigma\bar{\theta}, \qquad \theta' = \theta + \xi, \qquad \bar{\theta}' = \bar{\theta} + \bar{\xi}, \qquad z' = z + \theta\xi + \bar{\theta}\bar{\xi},$$

because the super Jacobian of the transformation is also 1. This may be seen from the transformation matrix $(1 + A)$, where

$$A = \begin{pmatrix} 0 & 0 & (\gamma^\mu \xi)_\alpha \\ 0 & 0 & \xi_\alpha \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $A^n = 0$, $n > 1$, we have $\mathrm{s\,det}\,(1 + A) \equiv \exp \mathrm{str} \ln (1 + A) = 1$.

We note the difference when we are considering field equations. In doing so we are interested in the functional derivative of the action with respect to the fields, therefore

$$\delta I = \int_\Gamma \frac{\delta \mathscr{L}}{\delta h} \cdot \delta h.$$

In this case the boundary is fixed because it is independent of the fields. A similar result will hold for all actions written as integrals over superspace based on the cone Γ.

The action (5.20) is a functional of the initial data of the differential constraints (5.18), (5.19). Following the formalism we have developed, it can be shown (see appendix) that the field equations are the correct on-shell equations for the hypermultiplet.

We are now able to construct an unconstrained superfield action, by following the Lagrange theorem of § 2. In doing so, one may divide the constraints (5.18), (5.19) into two sorts, one of which restricts solely the algebraic θ-structure of the superfields while the other restricts the z dependence of these fields, which of course is given by the spin-reducing condition, $\partial_5^2\phi = \Box\phi$. The Lagrange multiplier associated with this constraint has the usual boundary condition, while the Lagrange multipliers associated with the θ-constraints have the appropriate structure corresponding to an integral representation of $y^* \cdot h$ (we are using the notation of § 2) where y^* belongs to the dual-space of the Banach functional space Y. This approach is completely general and allows us to construct an unconstrained action for the constrained problem. The unconstrained fields associated with the z propagation are always fields on the vertex of the z cone.

One may also expect further simplifications for a given constrained action. Consider the case of the scalar field which we have analysed in § 2. We have shown that the variation with respect to the boundary fields gives solely the restriction

$$\lambda|_{x_0^5} = \lambda|_\infty. \tag{5.22}$$

Hence we can reformulate the problem as an unconstrained action

$$\int d\Omega\, dx^5 [u^T N u + \lambda (Mu - \partial_5 u)] \tag{5.23}$$

where now the functional space of the Lagrange multipliers satisfies (5.22). The field equations are those we have found, and the Lagrange multipliers are uniquely determined. It therefore turns out that the information contained in the boundary fields about the field equations is equivalent to a boundary condition on the Lagrange multipliers.

We are now able to formulate the constrained action (5.20) in the same way.

Consider as a first step the following action associated with the expression (5.20) expressed in terms of the boundary fields:

$$\begin{aligned}\int d^4x\, dx^5\, dx^6\, d^4\theta\, d^4\bar{\theta}\{&\Phi_i^+\Phi_i + \bar{K}^{\dot{\alpha}}_{ij}[\bar{D}^{0(i}_{\dot{\alpha}}(v_1^+ + u_1^+)^{j)} + \theta^{(i}_{\dot{\alpha}}(v_2^+ + u_2^+)^{j)}] \\ &+ K^{\alpha}_{ij}[D^{0(i}_{\alpha}(v_1^+ + u_1^+)^{j)} + \theta^{(i}_{\alpha}(v_2^+ + u_2^+)^{j)}] \\ &+ \mu^i(\partial_5 v_1^+ + v_2^+ - u_2^+)^i + \lambda^i \partial_6 \Phi^i + \mathrm{HC}\}\end{aligned} \tag{5.24}$$

where we have decomposed $\Phi = v_1 + u_1$, $\partial_5\Phi = v_2 + u_2$, where $v_1|_{x_0^5} = v_2|_{x_0^5} = 0$ and $u_1(x, \theta)$, $u_2(x, \theta)$ are the fields on the vertex of the z cone. We assume Φ, K, $\bar{K}$ and λ to be superfields and μ_1, μ_2 to have the transformation law

$$\delta\mu^i = -\delta(\theta K + \bar{\theta}\bar{K})^i \tag{5.25}$$

where $\delta\phi = \phi'(y') - \phi(y)$, and $y = (x, \theta, z)$. Under this assumption the action (5.24) is invariant under supersymmetric and central charge transformations. The field equations are

$$\Phi^i + (DK + \bar{D}\bar{K})^i - \partial_6 \lambda^i = 0, \tag{5.26}$$

$$\int_{x^5}\int_{x^6} \Phi^i + \left(D^0 \int_{x^5}\int_{x^6} K + \bar{D}^0 \int_{x^5}\int_{x^6} \bar{K}\right)^i = 0, \tag{5.27}$$

$$\mu^i = -(\theta K + \bar{\theta}\bar{K})^i, \tag{5.28}$$

(5.27) corresponds to the variation with respect to u_1^+, and the equation associated with respect to the variation of u_2^+ is redundant. We can see that (5.27) can be eliminated, after using (5.26), if we consider appropriate boundary conditions on λ^i, K_{ij}^{α} and $\bar{K}_{ij}^{\dot{\alpha}}$. We also notice that (5.28) determines only μ but does not give any further information about Φ and K, which are the fundamental geometrical objects.

We may now write the unconstrained action which follows from the above analysis,

$$I = \int_\Gamma d^4x\, d^4\theta\, d^4\bar{\theta}\, dx^5\, dx^6 (\Phi^{+i}\Phi^i + \bar{D}_{\dot{\alpha}}^{(i}\bar{K}_{ij}^{\dot{\alpha}}\Phi^{+j)} + D_{\alpha}^{(i}K_{ij}^{\alpha}\Phi^{+j)} + \lambda^i \partial_6 \Phi^i + \text{HC}) \tag{5.29}$$

where $K = K(x, \theta, x^5)$, and all the objects are superfields. The action is manifestly invariant under supersymmetry and central charge variations. We assume the functional space satisfies the boundary conditions $\lambda^i|_\infty = \lambda^i|_{x_0^6} = 0$, $K_{ij}^{\alpha}|_{x_0^5} = K_{ij}^{\alpha}|_\infty$, $\partial_5 \bar{K}_{ij}^{\dot{\alpha}}|_\infty = \partial_5 \bar{K}_{ij}^{\dot{\alpha}}|_{x_0^5}$, $\bar{K}_{ij}^{\dot{\alpha}}|_{x_0^5} = \bar{K}_{ij}^{\dot{\alpha}}|_\infty$, $\partial_5 K_{ij}^{\alpha}|_{x_0^5} = \partial_5 K_{ij}^{\alpha}|_\infty$. We notice that all these are covariant conditions, and also correspond to the component cases we considered in previous sections. The field equations are

$$\Phi^i + (DK + D\bar{K})^i = \partial_6 \lambda^i, \tag{5.30}$$

$$D_{\alpha(i}\Phi_{j)} = D_{\dot{\alpha}(i}\Phi_{j)} = 0, \qquad \partial_6 \Phi^i = 0. \tag{5.31), (5.32}$$

From (5.30) we get $\partial_6^2 \lambda^i = 0$, and from the boundary conditions $\lambda^i = 0$.

Let us define

$$\phi = \int_{x_0^5}^{\infty} dx^5\, \Phi_i.$$

From (5.30) and (5.31) we get

$$D^0{}_{\alpha(i}\phi_{j)} = -\int dx^5\, \theta_{\alpha(i}\partial_5 \Phi_{j)} = \int_{x_0^5}^{\infty} dx^5 \theta_{\alpha(i}\partial_5 (DK + \bar{D}\bar{K})_{j)} = 0 \tag{5.33a}$$

where we have used the boundary conditions for K. We also obtain

$$D^0_{\dot{\alpha}(i}\phi_{j)} = 0. \tag{5.33b}$$

We notice that these are covariant equations, as they should be. (5.33) implies $\Box \phi_j = 0$, therefore

$$\int dx^5\, \Box \Phi^j = \int dx^5\, \partial_5^2 \Phi_j = \partial_5 \Phi_j|_\infty - \partial_5 \Phi_j|_{x_0^5} = 0.$$

From the general on-shell boundary condition we have imposed on all of our formulation (see § 2)

$$\partial_5 \Phi_j|_\infty = 0. \tag{5.33c}$$

We obtain the correct on-shell field equation for the hypermultiplet.

We have thus succeeded in obtaining from the general Lagrange formulation, in terms of the boundary fields, an unconstrained full superspace action directly in terms of Φ, K and $\bar{K}$ with appropriate boundary conditions for the Lagrange multipliers. The superfield Φ satisfies the boundary condition (5.33*c*) we have imposed on all the formalism (see § 2) and essentially this is only a boundary condition for the on-shell fields to ensure central charges vanish on-shell.

We may include interaction of the hypermultiplet with the $N=2$ SYM gauge multiplet by modifying the constraints (5.18) so as to be gauge invariant by replacing D by the gauge-covariant derivative $\mathscr{D}_\alpha = D_\alpha - \mathrm{i}gA_\alpha$, where A_α is the SYM gauge potential. Thus (5.18) becomes (Sohnius 1978, Taylor 1980)

$$\mathscr{D}_{\alpha(i}\Phi_{j)} = \bar{\mathscr{D}}_{\dot{\alpha}(i}\Phi_{j)} = 0 \tag{5.34a}$$

and the total Lagrangian for the system will be

$$\int \mathrm{d}^4x\, \mathrm{d}x^5\, \mathrm{d}x^6\, d^4\theta\, \mathrm{d}^4\bar{\theta}[\mathrm{Tr}(F_{\alpha\beta})^2 + \Phi_i^+\Phi_i]. \tag{5.34b}$$

We may extend the above analysis both to other multiplets and to higher N. Thus for $N=4$ the fundamental multiplet is the superfield Φ_{ij} with constraints (Rands and Taylor 1983a, b)

$$D_{\alpha(i}\Phi_{jk)\boxplus} = \bar{D}_{\dot{\alpha}(i}\Phi_{jk)\boxplus} = 0 \tag{5.35}$$

(where raising and lowering of indices is performed by the USp(N) metric corresponding to a single central charge). Again we must take an irreducible representation, and in this case it is

$$\partial_n\Phi_{ij} = 0 \qquad (n = 6, 7, 8, 9, 10) \tag{5.36}$$

corresponding to the USp(4) metric chosen in (5.35). The constrained Lagrangian may be taken as

$$\int \mathrm{d}^4x\, \mathrm{d}x^5 \ldots \mathrm{d}x^{10}\, \mathrm{d}^8\theta\, \mathrm{d}^8\bar{\theta}\, \Phi_{ij}^+\Phi_{ij} \tag{5.37}$$

which gives the correct equations of motion on suitable variation as may be shown in a similar manner to the case for $N=2$. For $N=8$ the similar construction is in terms of the superfield Φ_{jklm} with constraints

$$D_{\alpha(i}\Phi_{jklm)\boxplus} = \bar{D}_{\dot{\alpha}(i}\Phi_{jklm)\boxplus} = 0 \tag{5.38}$$

with constraint

$$\partial_n\Phi_{jklm} = 0 \qquad (n = 6, \ldots, 18) \tag{5.39}$$

again singling out the single central charge assumed in (5.38), with associated USp(8) metric. The constrained Lagrangian will now be

$$\int \mathrm{d}^4x\, \mathrm{d}x^{14}\, \mathrm{d}^{16}\theta\, \mathrm{d}^{16}\bar{\theta}\, \Phi_{jklm}^+\Phi_{jklm} \tag{5.40}$$

which may be justified as before.

We have not considered the cases of more than one central charge in superfield form, though the component transformation rules and superfield constraints are known for this case (Bufton and Taylor 1983a). There are certain features about this case which require detailed analysis, which we leave to detailed consideration elsewhere.

However, we expect the same universal Lagrangian (5.37) or (5.40) for $N=4$ or 8, with constraints modified from (5.35), (5.36) or (5.38), (5.39) according to the specific number of these charges. Indeed it can only be in the constraint sector that the modifications to the theory can be inserted which determine the particular SUSY representation.

We may also wish to introduce interaction into (5.37) or (5.40). In the former this could be self-coupling or compensation of the $N=4$ SYM. The first of these possibilities appears difficult, as we noted in § 4. The second is, indeed, to be expected from the existence of the $N=3$ barrier (Rivelles and Taylor 1981, Taylor 1982b, Rocek and Siegel 1981), and particular compensating multiplets considered at the linearised level (Taylor 1982c). We expect that the total action will be of form (1.7) for $N=4$, with suitable field strength constraints appropriate to allowing the correct equations of motion to be obtained. Similar remarks apply to (5.40), with $N=8$ supergravity described by the Lagrangian (1.6) for $N=8$ and suitable torsion constraints. We hope to report on these constraints elsewhere.

6. Conclusions

We have shown that integration over central charges $x^5, \ldots, x^{2(N+1)}$ is to be interpreted as integration over the cone $\Gamma_N: x^i \geq x^i_{(0)}$, $5 \leq i \leq 2(N+1)$. (We have taken $x^i_{(0)}$ to be zero for each i, but this choice was for convenience). This interpretation was justified by showing that for each field theory of interest a constrained action, expressed as an integral over $\boldsymbol{R}^4 \times \Gamma_N$ or $S_{4+4N} \times \Gamma_N$ (where S_{4+4N} is the usual full superspace measure for N-SUSY without central charges) could be constructed so that the resulting equations of motion reduced to the expected four- (or $4+4N$)-dimensional ones for the boundary value fields at $x^i = x^i_{(0)}$. The form of the action was shown to be completely determined by the requirement of obtaining the correct field equations from a given set of constraints. In the superfield case the action was, in fact, unique on purely dimensional grounds, and the details of the field equations arose purely from the nature of the constraints.

Our interpretation of central charge dimensions seems very satisfactory from a physical point of view. There is no evidence available at present energies that space–time has more than four dimensions, and our construction of actions including central charge dimensions is such that this state of affairs is preserved. The interior of the cone Γ_N of the central charge variables is never directly observable.

The possibility of using a subspace of the θ-variables is not available to us for $N \geq 3$, since for such N there is no definition of chiral or similar subspaces without the associated disappearance of the central charges. Since the no-go theorems (Rivelles and Taylor 1981, Taylor 1982b, Rocek and Siegel 1981, Rivelles and Taylor 1983) show that *some* compensating multiplets with spin-reducing central charges are essential for the construction of 4-SYM and $N \geq 3$ SGR, a full geometric approach will have to allow central charges to appear somewhere and prevent the use of chirality (even if reduction of the number of θ-variables by a factor of two were considered possible, two central charge dimensions would still be needed for the definition of 4-SYM and 6 for 8-SGR). We thus have to use the central charge dimensions appropriate for description of the associated maximal use of integration over the θ-variables.

The extra central charge dimensions only arise as useful additional parameters for describing off-shell features. In particular they are essential in allowing the use of full integration over the Grassmann variables, θ, in superspace; without them the full θ-integration cannot be achieved.

We have seen that θ-space is not required when central charge variables are present. A component approach using central charge variables is quite satisfactory, though we saw that the superfield formulation, especially for the actions (1.6) and (1.7), were most elegant and simple. It is only in the constraints needed to be imposed on field strengths or torsions that we see complexity entering, although we still have a fully geometrical approach to the theory with these further features.

Our conclusion from our construction is that the use of Γ_N as the integration region for central charges allows a full geometrical interpretation to be given to N-SYM and N-SGR beyond the $N=3$ (Rivelles and Taylor 1981, Taylor 1982b) barrier. Similar cone interpretations should be valid for central charges needed to penetrate the $N=2$ barrier existing in higher-dimensional supersymmetric theories (Rivelles and Taylor 1983). We are thus prepared to construct 4-SYM and 8-SGR with maximal supersymmetry.

It is appropriate to relate our work to the recent discussion of Rogers (1982), where a constrained action, involving integration over two central charge dimensions, was proposed for $N=2$ SGR. This was claimed to give the correct dynamics by using the method of Wess and Zumino (1978), without need to restrict the central charge integration region in any way. If the constraints given in Rogers (1982) lead to the Dirac equation (1.4) then in order to obtain the field equations it is necessary to specify the boundary conditions at some lower-dimension surface in order that the field equations (for the boundary values) may then be derived. The position of this surface is not of any relevance, as long as it is chosen to have the correct dimensions (less than the full Bose dimensions by the number of central charges). Thus the unconstrained action is to be regarded as an integral over some region Γ in central charge directions, with the four-dimensional surface $V(\Gamma)$ identified with $\boldsymbol{R}^4$. We have chosen Γ to be the cone $(x^5 \geq x_0^5, x^6 \geq x_0^6, \ldots)$, with $V(\Gamma) = (x^5 = x_0^5, \ldots)$; the precise nature of Γ is clearly not important. On the other hand if the constraints do not lead to the Dirac equation (1.4) then they cannot help to broach the $N=3$ barrier, and that approach will have to be modified along the lines we present here.

Our construction has been purely at the classical level. We are naturally interested in the quantum features of our theories. We might suspect from our unconstrained Lagrangians that they are the full central charge superspace equivalents of actions written in first-order form

$$A = \int (p\dot{q} - H(q,p))\,\mathrm{d}t \qquad (6.1)$$

where in our case p is a set of Lagrange multipliers and q are superfields such as the hypermultiplet Φ_i in (5.18). Quantisation may be performed as usual by taking a 'sum over paths' formulation with density $\mathrm{e}^{\mathrm{i}A}$; this can be reduced to the usual functional integral over the fields q after integration over p has been performed. It is not necessary to perform such p-integration, and in our case we do not wish to. Thus our expected quantum features are to be derived from

$$G(J,K) = \int \mathrm{d}[\lambda]\,\mathrm{d}[\Phi]\exp\{\mathrm{i}[A(\Phi,\lambda) + \Phi J + \lambda K]\} \qquad (6.2)$$

where J and K are external sources coupled to the basic superfields Φ and Lagrange multiplier fields λ. We may conjecture that the uniqueness of the actions (1.6) and

(1.7) can allow, at most, coupling constant renormalisation effects on-shell, since the only counter-terms not vanishing on shell and defined over the whole central-charge superspace are proportional to the original Lagrangians. The detailed mechanism of such quantisation has still to be developed. We propose to analyse this possibility and the more general nature of the resulting quantum mechanics elsewhere.

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Appendix

We are going to prove that the field equations determined by (5.20) are

$$\partial_5\Phi_i = \partial_6\Phi_i = 0. \tag{A1}$$

(5.18) implies

$$\partial_5^2\Phi_i = \Box\Phi_i; \tag{A2a}$$

we also have

$$\partial_6\Phi_i = 0. \tag{A2b}$$

Let us define $A_i^1(x^5, x^6)$, $A_i^2(x^5, x^6)$ the two independent modes which solve (A2). It is straightforward to prove that these modes satisfy

$$D^0_{\alpha(i}A^1_{j)} + \theta_{\alpha(i}A^2_{j)} = 0, \qquad D^0_{\alpha(i}A^2_{j)} + \theta_{\alpha(i}\Box A^1_{j)} = 0 \tag{A3a, b}$$

and the corresponding constraints for the other chirality. We can express (5.20) in the following way

$$I = \int d^4x\, d^4\theta\, d^4\bar\theta\, dx^5\, dx^6 (A_i^1 + ZA_i^2)^+(A_i^1 + ZA_i^2)^+ \tag{A4}$$

A_i^1, A_i^2 satisfying (A2) and (A3). (A2) and (A3) are equivalent to (5.18). For fixed x^5, x^6 let us integrate in θ, $\bar\theta$ using (5.3). We get

$$I = \int d^4x\, dx^5\, dx^6\, \mathscr{L}(A_i^2|_{\theta=0}, A_i^2|_{\theta=0}, \phi_i|_{\theta=0}, \psi_i|_{\theta=0}) \tag{A5}$$

where

$$A_i^2(x^5, x^6)|_{\theta=0}, \qquad A_i^2(x^5, x^6)|_{\theta=0}, \qquad \phi_i(x^5, x^6)|_{\theta=0}, \qquad \psi_i(x^5, x^6)|_{\theta=0}$$

satisfy constraints (A2). We show elsewhere (Hassoun *et al* 1983) that I is the action we have already studied in the component approach. Therefore the stationary points of (A5) and thus of (5.20) satisfy (A1).

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